Recent Progress on Amalgamation

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Mathematics Colloquium
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18 March 2024

Part I:
Some Background

The idea of amalgamation

Some familiar groups:

*	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

*	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

The idea of amalgamation

First attempt to embed both groups in a bigger group:

*	0	1	2	a	b	c
0	0	1	2	a	b	c
1	1	2	0			
2	2	0	1			
a	a			0	c	b
b	b			c	0	a
c	c			b	a	0

Turns out to be impossible on 6 elements...

BUT you can embed both groups in their direct product.

Amalgamation formally

Definition:

Suppose that K is a class of algebraic structures. We say that K has the strong amalgamation property if whenever $\mathbf{B}, \mathbf{C} \in \mathsf{K}$ intersect on a common substructure $\mathbf{A}$, there exists $\mathbf{D} \in \mathsf{K}$ that contains $\mathbf{B}$ and $\mathbf{C}$ as substructures.

Definition:

Suppose that K is a class of algebraic structures. A span in K is a pair of injective homomorphisms $\langle f_{\mathbf{B}} \colon \mathbf{A} \to \mathbf{B}, f_{\mathbf{C}} \colon \mathbf{A} \to \mathbf{C} \rangle$, where $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathsf{K}$.

An amalgam of the span $\langle f_{\mathbf{B}} \colon \mathbf{A} \to \mathbf{B}, f_{\mathbf{C}} \colon \mathbf{A} \to \mathbf{C} \rangle$ is a pair of injective homomorphisms $\langle g_{\mathbf{B}} \colon \mathbf{B} \to \mathbf{D}, g_{\mathbf{C}} \colon \mathbf{C} \to \mathbf{D} \rangle$ such that $g_{\mathbf{B}} \circ f_{\mathbf{B}} = g_{\mathbf{C}} \circ f_{\mathbf{C}}$, and an amalgam is said to be in K when further $\mathbf{D} \in \mathsf{K}$.

The class K has the amalgamation property if every span in K has an amalgam in K.

Some classical results

- The class of all groups has the strong amalgamation property (Schreier 1927).
- Abelian groups also have the strong amalgamation property.
- Semigroups do not even have the amalgamation property (Kimura 1957), and neither do solvable groups (Neumann 1960).
- Fields (Jónsson 1965) and integral domains (Cornish 1977) both have the AP.
- Topological spaces and even Stone spaces have the AP, but Hausdorff abelian groups don't (Tholen 1982).

Setting the stage

Implication from these examples: Not much of a general picture yet. But we can do better by zeroing in on certain classes of structures.

Definition:

By an algebra we mean a set with some operations (each with some given arity).

A variety is a class of algebras that are defined by equations.

Note: Varieties are the same as classes that are closed under taking homomorphic images ($\mathbb{H}$), subalgebras ($\mathbb{S}$), and direct products ($\mathbb{P}$).

Congruences

If **A** is an algebra, a congruence of **A** is just an equivalence relation Θ that is compatible with all the operations of **A**: If f is an n-ary operation of **A** and $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$,

$$a_1 \Theta b_1, a_2 \Theta b_2, \ldots, a_n \Theta b_n \Rightarrow f(a_1, \ldots, a_n) \Theta f(b_1, \ldots, b_n).$$

Congruences of groups are given by normal subgroups and congruences of rings are given by ideals, for example. But for semigroups and other algebras one must consider binary relations.

Note: For any algebra, congruences are the same as kernals of homomorphisms:

$$ker(h) = \{(x, y) \in A^2 \mid h(x) = h(y)\}.$$

Ordered algebras

Definition:

A lattice is an algebra of the form $(A, \wedge, \vee)$, where $\wedge$ and $\vee$ are binary operations that satisfy:

$$\begin{array}{cc} x \wedge x = x & x \vee x = x \\ x \wedge y = y \wedge x & x \vee y = y \vee x \\ x \wedge (y \wedge z) = (x \wedge y) \wedge z & x \vee (y \vee z) = (x \vee y) \vee z \\ x \wedge (x \vee y) = x & x \vee (x \wedge y) = x \end{array}$$

Lattices turn out to be partially ordered: One can always define $x \leq y$ iff

$$x \wedge y = x$$

.

In fact, lattices are exactly partially ordered sets where

$$x \wedge y = inf\{x, y\} \text{ and } x \vee y = sup\{x, y\}$$

.

Ordered algebras

Definition:

A Boolean algebra is an algebra of the form $(A, \wedge, \vee, ', 0, 1)$, where $(A, \wedge, \vee)$ is a lattice with least element 0 and greatest element 1, and

$$x \wedge x' = 0 \text{ and } x \vee x' = 1.$$

Boolean algebras are familiar as fields of sets, and from this it's clear that they're distributive lattices:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

Definition:

A Heyting algebra is an algebra of the form $(A, \wedge, \vee, \to, 0, 1)$, where $(A, \wedge, \vee)$ is a lattice with least element 0 and greatest element 1, and for all $x, y \in A$,

$$x \wedge y \leq z \iff z \leq y \to z.$$

Remarkably, Heyting algebras form a variety too. Boolean algebras are equivalent to Heyting algebras that satisfy

$$x \vee (x \to 0) = 1.$$

(Think of $x \to 0$ as x').

Amalgamation in ordered algebras

- Boolean algebras have the AP (Dwinger-Yaqub 1963).
- Lattices (Jónsson 1960) and distributive lattices (Pierce 1968) have the AP, but they are the only non-trivial varieties of lattices that do (Grätzer-Jónsson-Lakser 1973).
- The variety of all Heyting algebras have the AP (Day, around 1970), and are one of only 8 varieties of Heyting algebras that do (Maksimova 1977).

One reason to tackle these kinds of ordered algebras is that they inspire general principles for thinking about amalgamation, but another is that they are necessary to phrase these principles.

Note that if $\mathbf{A}$ is any algebra, then it's collection of congruences $\operatorname{Con}\mathbf{A}$ is a lattice when they are ordered by $\subseteq$ (considered as sets of ordered pairs).

Definition:

An algebra $\mathbf{A}$ is called congruence distributive if $\operatorname{Con}\mathbf{A}$ is distributive. A class K of algebras is congruence distributive if each of its members is.

Lattices, distributive lattices, Heyting algebras, and Boolean algebras are all examples of congruence distributive varieties.

Congruence extension property

Definition:

Let $\mathbf{A}$ be an algebra. Then $\mathbf{A}$ is said to have the congruence extension property if for each subalgebra $\mathbf{B}$ of $\mathbf{A}$ and each congruence Θ of $\mathbf{B}$, there exists a congruence Ψ of $\mathbf{A}$ such that $\Theta = \Psi \cap B^2$. A class K of algebras has the congruence extension property if all of its members do.

Lattices, distributive lattices, Heyting algebras, Boolean algebras all have the congruence extension property.

Part II:
Tools and Techniques

Reduction to generating classes

Amalgamation (for a given class) can be very effectively studied when it can be reduced to a more easily handled class of algebras.

Definition:

An algebra $\mathbf{A}$ is called finitely subdirectly irreducible (FSI) if the least congruence $\Delta = \{(x, y) \mid x = y\}$ cannot be written as the intersection of two other congruences. It is subdirectly irreducible if Δ cannot be written as any intersection of other congruences.

The only non-trivial finitely subdirectly irreducible Booolean algebra is the 2-element one, a Heyting algebra is FSI if its top elements can't be written as a join of two other elements, etc.

Some sufficient conditions

Theorem (Metcalfe-Montagna-Tsinakis 2014):

Let K be a subclass of a variety V satisfying

1. K is closed under isomorphisms and subalgebras;
2. every subdirectly irreducible member of V belongs to K;
3. for any $\mathbf{B} \in \mathsf{V}$ and subalgebra $\mathbf{A}$ of $\mathbf{B}$, if $\Theta \in \mathrm{Con}\,\mathbf{A}$ and $\mathbf{A}/\Theta \in \mathsf{K}$, then there exists a $\Phi \in \mathrm{Con}\,\mathbf{B}$ such that $\Phi \cap A^2 = \Theta$ and $\mathbf{B}/\Phi \in \mathsf{K}$;
4. every span of finitely generated algebras in K has an amalgam in V.

Then V has the amalgamation property.

For the strong AP

Theorem (F.-Galatos 2024):

Let V be an arithmetical variety and V_{FSI} be its class of finitely subdirectly irreduicble members. Suppose:

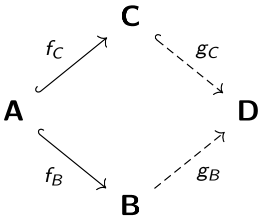
1. V_{FSI} is a universal class;
2. for any $\mathbf{B} \in V$ and subalgebra $\mathbf{A}$ of $\mathbf{B}$, if $\Theta \in \mathrm{Con}\,\mathbf{A}$ and $\mathbf{A}/\Theta \in V_{FSI}$, then there exists a $\Phi \in \mathrm{Con}\,\mathbf{B}$ such that $\Phi \cap A^2 = \Theta$ and $\mathbf{B}/\Phi \in V_{FSI}$;
3. every span in V_{FSI} has a strong amalgam in V.

Then V has the amalgamation property.

One-Sided Amalgamation

Definition:

Let K be a class of algebraic structures. We say that K has the one-sided amalgamation property (or 1AP) if for every span $\langle f_B\colon \mathbf{A} \to \mathbf{B}, f_C\colon \mathbf{A} \to \mathbf{C}\rangle$ in K there exists $\mathbf{D} \in$ K, an embedding $g_B\colon \mathbf{B} \to \mathbf{D}$, and a homomorphism $g_C\colon \mathbf{C} \to \mathbf{D}$ such that $g_B \circ f_B = g_C \circ f_C$.



Characterizing the AP

Theorem (F.-Metcalfe 2024):

Let V be any variety with the congruence extension property such that V_{FSI} is closed under subalgebras. Then following are equivalent:

1. V has the amalgamation property.
2. V has the one-sided amalgamation property.
3. V_{FSI} has the one-sided amalgamation property.
4. Every span in V_{FSI} has an amalgam in $V_{FSI} \times V_{FSI}$.
5. Every span of finitely generated algebras in V_{FSI} has an amalgam in V.

Note: (4) and (5) provide some effective bounds on searches for amalgams.

Decidability

Definition:

A variety finitely generated if it is generated as a variety by some given finite set of finite algebras of finite signature.

Theorem (F.-Metcalfe 2024):

Let V be a finitely generated congruence-distributive variety such that V_{FSI} is closed under subalgebras. There exists an effective algorithm to decide if V has the amalgamation property.

Old results made easy

Theorem (Dwinger-Yaqub 1963):

The variety of Boolean algebras has the amalgamation property.

Proof: The only FSI Boolean algebras are the 1-element one and the 2-element one, and the only spans you can make from these trivially have one-sided amalgams. $\square$

Extensibility and simple algebras

Definition:

We say that an algebra $\mathbf{A}$ is extensible if whenever $\mathbf{B}, \mathbf{B}'$ are subalgebras of $\mathbf{A}$ and $h\colon \mathbf{B} \to \mathbf{B}'$ is an isomorphism, there exists an automorphism $\hat{h}\colon \mathbf{A} \to \mathbf{A}$ extending h.

Theorem (F.-Metcalfe 2024+):

Let $\mathbf{A}$ be a finite simple algebra with CEP, and assume that the variety it generates $\mathsf{V} = \mathbb{V}(\mathbf{A})$ is congruence distributive. Further assume that $\mathbf{A}$ has no 1-element subalgebras. Then V has the amalgamation property if and only if $\mathbf{A}$ is extensible.

Heuristic Approaches

The complexity of the decision procedure is quite bad and we often want to use computational resources for varieties that aren't finitely generated. In these cases, some heuristic approaches are available. Basic recipe:

1. Enumerate small triples $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ with $\mathbf{A} \leq \mathbf{B}$ and $\mathbf{A} \leq \mathbf{C}$.
2. Write down the atomic diagrams of **B**, **C** with appropriate names for $A = B \cap C$.
3. Feed the resulting constraints into a prover/model builder.
4. Simultaneously run a search for models (= amalgamable span) and proofs of falsum (= proof that the span cannot be amalgamated).

This is a necessarily interactive approach: Need to know something about the algebras to optimize computation.

Essential extensions

There are some other formulations of the amalgamation property that are especially useful for studying algebraic structures whose FSIs have a particular form.

Definition:

Let $\mathbf{A}, \mathbf{B}$ be algebras. An injective homomorphism $\varphi\colon \mathbf{A} \to \mathbf{B}$ is called an essential embedding if every congruence Θ of $\mathbf{B}$, $\Theta \neq \Delta_B$ implies $\Theta \cap \varphi[A]^2 \neq \Delta_{\varphi[A]}$. A span $\langle f_B\colon \mathbf{A} \to \mathbf{B}, f_C\colon \mathbf{A} \to \mathbf{C}\rangle$ is essential if f_C is essential, and a class K has the essential AP if every essential span in K has an amalgam in K.

Theorem (F.-Santschi 2024+):

Let V be a variety with the CEP such that $\mathbb{HS}(\mathsf{V}_{\mathsf{FSI}}) = \mathsf{V}_{\mathsf{FSI}}$. Then V has the AP if and only if $\mathsf{V}_{\mathsf{FSI}}$ has the essential AP.

Part III: A Case Study

Triangular norms

A triangular norm (or t-norm) is a commutative and associative binary operation $\cdot$ on $[0,1]$ that has 1 as an identity element.

Every continuous t-norm $\cdot$ has an associated residual operations, i.e binary operations $\to$ on $[0,1]$ such that

$$x \cdot y \leq z \iff x \leq y \to z.$$

The meet $\wedge$ and join $\vee$ are definable from $\cdot$ and $\to$ alone, and such structures have often been taken as central models for fuzzy logics.

BL-algebras

Definition:

A BL-algebra is an algebra $(A, \wedge, \vee, \cdot, \to, 0, 1)$ such that $(A, \wedge, \vee)$ is a lattice with bounds 0 and 1, $(A, \cdot, 1)$ is a monoid, $\to$ is the residual of $\cdot$, and

$$x \cdot (x \to y) = x \wedge y \text{ and } (x \to y) \vee (y \to x) = 1.$$

BL-algebras were introduced by Hájek in 1999, and are notable because they axiomatize the equational theory of continuous t-norms: BL-algebras are exactly the variety of algebras generated by continuous t-norms on $[0, 1]$.

A bit about structure

The structure of BL-algebras is very well described in terms of so-called ordinal sums.

The details of this are technical, but in particular, FSI BL-algebras are totally ordered and closed under homomorphic images and subalgebras.

Amalgamation in BL-algebras is thus very much amenable to study via the essential AP.

Interpolation

In fact, amalgamation has been scrutinized quite a bit in BL-algebras because of its connection to interpolation.

Definition:

- For any class K of BL-algebras, define equational consequence, for a set of equations $E \cup \{u \approx w\}$, by $E \models_{\mathsf{K}} u \approx w$ if and only if

  For each $\mathbf{A} \in \mathsf{K}$ and each assignment h into $\mathbf{A}$,

  $h(u) = h(w)$ whenever $h(s) = h(t)$ for all $(s \approx t) \in E$.

- A variety V of BL-algebras has the equational interpolation property if whenever $E \cup \{\epsilon\}$ is a set of equations with $E \models_{\mathsf{V}} \epsilon$, there exists an equation δ whose variables are among those appearing in both E and ϵ such that $E \models_{\mathsf{V}} \delta$ and $\delta \models_{\mathsf{V}} \epsilon$.

Amalgamation and interpolation in BL-algebras

- It is well-known that a variety of BL-algebras has interpolation iff it has the AP (true much more generally).
- Because of this, there is quite some history of studying amalgamation in varieties of AP in order to get at interpolation for fuzzy logics:
  - Montagna (2006) proves AP for variety of all BL-algebras along with many natural subvarieties, but gives uncountably many subvarieties without AP.
  - Subsequent partial classifications by Aguzzoli and Bianchi (2021 and 2023).
  - Main open question: Complete classification. How many varieties of BL-algebras with AP are there (countable or uncountable)?

A classification for BL-algebras

Using essential AP:

Theorem (F.-Santschi 2024+):

There are only countably many varieties of BL-algebras with the amalgamation property.

In fact, these can be explicitly described in terms of their generating algebras and partitioned into a countably infinite family of finite intervals in the lattice of all subvarieties of BL-algebras.

Thank you!