Poset Products and Strict Implication

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Strict implication classically

Strict implication arises from prefixing material implication by a modal necessity operator: $\Box(\varphi \to \psi)$.

- Strict implication was Lewis's original motivation for studying modal logic, and kicked off the modern era of the subject.
- In Kripke frames, each world is endowed with classical logic and locally we have material implication.
- Modal logics above S4 have an especially nice strict implication, corresponding (via Gödel's translation) to intuitionistic logic.
- Today's talk: Effort to understanding which substructural logics are logics of strict implication. Focus is on logics without contraction, i.e., $\Gamma, \varphi, \varphi \vdash \Sigma \ / \ \Gamma, \varphi \vdash \Sigma$.

Residuated lattices

Definition:

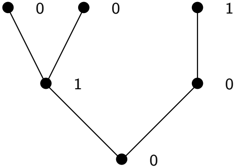
A (bounded, commutative, integral) residuated lattice is an algebra $(A, \wedge, \vee, \cdot, \rightarrow, 0, 1)$ such that

- $(A, \wedge, \vee, 0, 1)$ is a bounded lattice.
- $(A, \cdot, 1)$ is a commutative monoid.
- For all $x, y, z \in A$,

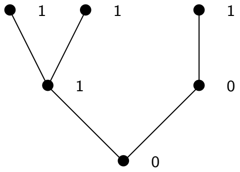
$$x \cdot y \leq z \iff x \leq y \rightarrow z.$$

Residuated lattices give the equivalent algebraic semantics for extensions of the Full Lambek calculus (with exchange and weakening).

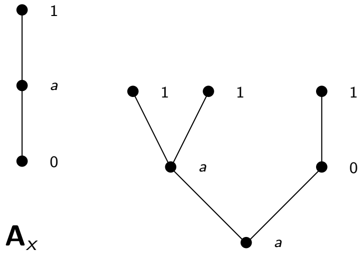
Strict implication and the heredity condtion



Strict implication and the heredity condtion



Algebra valued frames and models

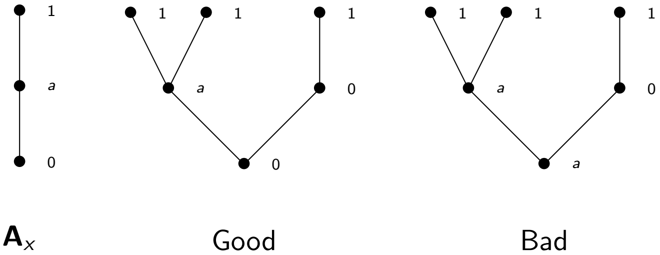


Antichain labelings

Definition:

Let $(X, \leq)$ be a poset, and let $\{\mathbf{A}_x : x \in X\}$ is an indexed collection of residuated lattices sharing a common least element 0 and common greatest element 1. An antichain labeling is a choice function $f \in \prod_{x \in X} A_x$ such that for all $x, y \in X$,

$$x < y \implies f(x) = 0 \text{ or } f(y) = 1.$$



Poset products

Definition:

Let $(X, \leq)$ be a poset and let $\{\mathbf{A}_x : x \in X\}$ is an indexed collection of residuated lattices sharing a common least element 0 and greatest element 1. Set $B = \{f \in \prod_{x \in X} : f$ is an ac-labeling$\}$. We define operations in B as follows. The operations $\wedge, \vee, \cdot, 0, 1$ are defined pointwise, and the operation $\to$ is defined by

$$(f \to g)(x) = \begin{cases} f(x) \to_x g(x) & \text{if for all } y > x, f(y) \leq_x g(y) \\ 0 & \text{otherwise.} \end{cases}$$

The algebra $\mathbf{B}$ with these operation is called the poset product.

Poset products as conuclear images

Let $(X, \leq)$ be a poset and $\{\mathbf{A}_x : x \in X\}$ is an indexed collection of residuated lattices sharing a common least element 0 and common greatest element 1. Set $\mathbf{B} = \prod_{x \in X} \mathbf{A}_x$ and define a map $\Box\colon B \to B$ by

$$\Box(f)(x) = \begin{cases} f(x) & \text{if } f(y) = 1 \text{ for all } y > x \\ 0 & \text{if there exists } y > x \text{ with } f(y) \neq 1. \end{cases}$$

Then $\Box$ is a conucleus on the direct product. The conuclear image coincides with the poset product:

$$\mathbf{B}_\Box = \prod_{(X,\leq)} \mathbf{A}_x.$$

The benefits of poset products

- Realizing the algebras in a variety of residuated lattices as (embedded into) poset products also realizes them as algebras of strict implication, i.e. the implication is the boxed implication with respect to some S4-type box operator.
- This has been used to give modal translations for some prominent substructural logics, e.g. the logic corresponding to GBL-algebras.
- Also useful for giving relational semantics for substructural logics.

What can we say about poset products?

We will start with a poset product

$$\mathbf{B} = \prod_{(X,\leq)} \mathbf{A}_x$$

and explore what must be true of it. The first point to notice is that the Heyting algebra of up-sets $\mathrm{Up}(X, \leq)$ always embeds in $\mathbf{B}$; if $U \in \mathrm{Up}(X, \leq)$ then

$$\chi_U(x) = \begin{cases} 1 & \text{if } x \in U, \\ 0 & \text{if } x \notin U. \end{cases}$$

In fact, this gives $\mathrm{Up}(X, \leq)$ as a complete perfect Heyting subalgebra of $\mathbf{B}$.

What can we say about poset products?

Write $H_{\mathbf{B}}$ for the complete Heyting subalgebra of $\mathbf{B}$. An easy calculation shows that if $i \in H_{\mathbf{B}}$ and $f \in \mathbf{B}$, then because products and meets are computed pointwise

$$i \cdot f = i \wedge f.$$

Also, there is always a least element of $H_{\mathbf{B}}$ above an element of $\mathbf{B}$ and dually:

$$f^{\uparrow} = \bigwedge\{i \in H_{\mathbf{B}} \mid f \leq i\},$$

$$f^{\downarrow} = \bigvee\{i \in H_{\mathbf{B}} \mid i \leq f\}.$$

What can we say about poset products?

Now for any $f \in \mathbf{B}$, the set $\{x \in X \mid f(x) \neq 0, 1\}$ forms an antichain that we denote by S_f. We can always define an antichain labeling for a given $p \in S_f$ by

$$f_p(x) = \begin{cases} 1 & \text{if } f(x) = 1, \\ f(x) & \text{if } x = p, \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that $f = \bigvee_{p \in S_f} f_p$.

Definition:

Suppose $\mathbf{A}$ is a residuated lattice with a perfect complete Heyting subalgebra $H_{\mathbf{A}}$. We say that $x \in \mathbf{A}$ is principal if $x^{\uparrow}$ is a completely join-irreducible element of $H_{\mathbf{A}}$.

What can we say about poset products?

The elements f_p that give us the representation of $f \in \mathbf{B}$ as a join of principals also satisfy another property: They are disjoint.

Definition:

Two principal elements x, y of $\mathbf{A}$ are said to be disjoint if $x^{\uparrow} \neq y^{\uparrow}$.

One can show that in any poset product, any join of disjoint principal elements exists.

What can we say about poset products?

There is one other notable way that $H_{\mathbf{B}}$ interacts with $\mathbf{B}$:

Proposition:

Suppose that $i \in H_{\mathbf{B}}$ and $f \in \mathbf{B}$. Then there exists $j \in H_{\mathbf{B}}$ such that $i \wedge j \leq f \leq i \vee j$.

We can abstract the properties we have just identified:

Definition:

A centered residuated lattice is a pair $\langle \mathbf{A}, H_{\mathbf{A}} \rangle$ such that:

1. $\mathbf{A}$ is a bounded, commutative, integral residuated lattice.
2. $H_{\mathbf{A}}$ is a perfect Heyting algebra and a complete subalgebra of $\mathbf{A}$ such that $ax = a \wedge x$ for all $a \in H_{\mathbf{A}}$, $x \in A$.
3. Each element of $\mathbf{A}$ is a join of principal elements, and the join of each collection of disjoint principal elements of $\mathbf{A}$ exists.
4. For each $a \in H_{\mathbf{A}}$ and $x \in A$, there exists $b \in H_{\mathbf{A}}$ such that $a \wedge b \leq x \leq a \vee b$.

A representation theorem

Theorem:

Let $\langle \mathbf{A}, \mathbf{H_A} \rangle$ be a centered residuated lattice and let $(J^\infty(H_\mathbf{A}), \geq)$ be the poset of completely join-irreducible elements of $H_\mathbf{A}$. Then $\mathbf{A} \cong \prod_{(J^\infty(H_\mathbf{A}), \geq)} \mathbf{A}_x$ for quotients $\mathbf{A}_x$ in the signature $\{\wedge, \vee, \cdot, 0, 1\}$.

Applications

- Particularly useful when $H_{\mathbf{A}}$ is definable from the residuated lattice signature alone (e.g. when all idempotents are in $H_{\mathbf{A}}$).
- Of course, what we'd really like to think about are weakly centered residuated lattices: Those that embed into centered residuated lattices.
- This can be thought of as a kind of completion-like construction that adds joins of disjoint principal elements.

Applications

For each $n \in \mathbb{N}$, let S_n denote the subvariety of residuated lattices axiomatized by:

1. $a^n b = a^n \wedge b$.
2. $a^n \to b^n = (a^n \to b^n)^2$.
3. $a \leq b^n \vee (b^n \to a^n)$.

Further, for each $n \in \mathbb{N}$ denote by C_n the subvariety of S_n axiomatized by $(a \to b) \to (b \to a) = b \to a$.

Theorem

Each member of S_n is weakly centered, hence embeds into a poset product of simple n-potent residuated lattices. Each member of C_n embeds into a poset product of simple n-potent residuated chains.

Thank you!