Toward a Systematic Theory of Amalgamation

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Amalgamation and Interpolation

Amalgamation is (at least for me) interesting because of its close relationship to interpolation.

Interpolation refers to a cluster of metalogical properties asserting that if A entails B, then there exists I dealing only with the subject matter common to A and B and such that

$$A \text{ entails } I \text{ and } I \text{ entails } B.$$

I is called an interpolant and gives a kind of explanation for why A entails B.

The intuitive notions ‘entailment’ and ‘common subject matter’ are cashed out in many different ways.

Interpolation: Some Variants

Craig interpolation:

$$A \to B \Rightarrow \exists I(\mathrm{var}(I) \subseteq \mathrm{var}(A) \cap \mathrm{var}(B), A \to I, I \to B)$$

Deductive interpolation:

$$A \vdash B \Rightarrow \exists I(\mathrm{var}(I) \subseteq \mathrm{var}(A) \cap \mathrm{var}(B), A \vdash I, I \vdash B)$$

Uniform interpolation: For any $S \subseteq \mathrm{var}(A)$, there exists I, $\mathrm{var}(I) \subseteq S$, so that if $\mathrm{var}(A) \cap \mathrm{var}(B) \subseteq S$,

$$A \to B \iff I \to B$$

Interpolation: Applications

- Philosophical issues (e.g. argumentation theory)
- Hardware and software verification (Ken McMillan approach)
- Database theory (primarily uniform interpolation)
- Recent efforts to use interpolants as a resource for SMT solving.

Toward a coherent picture

Interpolation turned out to be a treacherous subject. There are many published proofs by well known researchers that were later found to be wrong, leaving the result open, with neither proofs nor counter-examples. Although in general we have some ideas of how and why interpolation may hold or fail for a given system, we still have the persistent feeling that really we need to obtain our results (if we can) logic by logic, case by case, and that a slight variation in the logic may change the outcome. Yet we feel that interpolation is a coherent research area and we just have to wait and see to figure out what is going on.

-D. Gabbay and L. Maksimova,
Interpolation and Definability.
Oxford University Press, 2005.

Interpolation: A Short Bibliography

This talk is largely based on:

- W. Fussner and N. Galatos, Semiconic idempotent logic I: Structure and local deduction theorems. Ann. Pure Appl. Logic 175, 103443 (2024). `https://doi.org/10.1016/j.apal.2024.103443`
- W. Fussner and N. Galatos, Semiconic idempotent logic II: Beth definability and deductive interpolation. To appear in Ann. Pure Appl. Logic. `https://arxiv.org/abs/2208.09724v3`.
- W. Fussner and G. Metcalfe, Transfer theorems for finitely subdirectly irreducible algebras. J. Algebra 640, pp. 1-20 (2024). `https://doi.org/10.1016/j.jalgebra.2023.11.003`.

Interpolation: A Short Bibliography

- W. Fussner, G. Metcalfe, and S. Santschi, Interpolation and the Exchange rule. https://arxiv.org/abs/2310.14953
- W. Fussner and S. Santschi, Interpolation in Linear Logic and Related Systems. https://arxiv.org/abs/2305.05051
- W. Fussner and S. Santschi, Interpolation in Hájek's Basic Logic. https://arxiv.org/abs/2403.13617
- W. Fussner, Revisiting Interpolation in Relevant Logics. Manuscript.
- W. Fussner and S. Santschi, Amalgamation in Semilinear Residuated Lattices. Manuscript.

Part I:
Tools and Techniques

Proof-theoretic Techniques

- Probably any discussion of interpolation should begin with Maehara's method.
- Available when a suitable analytic sequent calculus (or sometimes tableau system) is available.
- Essentially operates by iteratively partitioning pairs of sets of formulas in two parts, with a variable in common.
- When successful, results in a (constructively produced!) Craig interpolant.
- .... But sharply limited by the existence of a good proof theory (see R. Jalali and A. Tabatabai, Universal Proof Theory: Semi-analytic Rules and Craig Interpolation, `https://arxiv.org/abs/1808.06256`).

Theorem (Czelakowski-Pigozzi 1999):

Let $\vdash$ be an algebraizable deductive system with a local deduction theorem whose equivalent algebraic semantics is the variety $\mathcal{V}$. Then $\vdash$ has the deductive interpolation property if and only if $\mathcal{V}$ has the amalgamation property.

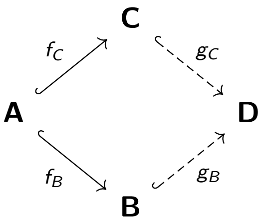
Major question: How to establish/refute the AP.

Answer: Reduce the complexity of the problem to some tractable generating class.

Amalgamation

Definition:

Let K be a class of algebraic structures. A span in K is a quintuple (A, B, C, f_B, f_C), where $A, B, C \in$ K and $f_B\colon A \to B$, $f_C\colon A \to C$ are embeddings. We say that K has the amalgamation property (or AP) if for every span (A, B, C, f_B, f_C) in K there exists $D \in$ K and embeddings $g_B\colon B \to D$ and $g_C\colon C \to D$ such that $g_B \circ f_B = g_C \circ f_C$. K has the strong AP if we can take $g_B[B] \cap g_C[C] = g_C \circ f_C[A]$.



Some familiar groups:

*	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

*	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

The idea of amalgamation

First attempt to embed both groups in a bigger group:

*	0	1	2	a	b	c
0	0	1	2	a	b	c
1	1	2	0			
2	2	0	1			
a	a			0	c	b
b	b			c	0	a
c	c			b	a	0

Turns out to be impossible on 6 elements...

BUT you can embed both groups in their direct product.

Finitely generated algebras are enough

Theorem (F.-Metcalfe 2024)

Let K be a universal class of algebras such that every span of finitely generated algebras in K has an amalgam in K. Then K has the amalgamation property.

Proof: Let (A, B, C, f_B, f_C) be a span in K, assuming WLOG that f_B and f_C are just inclusion maps.

Let Σ be the union of the theory of K with the atomic diagrams of B and C. Then the span has an amalgam iff Σ has a model.

Now consider Σ' to be the theory of K with any finite subset of the atomic diagrams of B and C, giving respective subalgebras A', B', C' generated by the finitely many elements named in Σ'.

By assumption (A', B', C', f'_B, f'_C) has an amalgam in K. So Σ has a model by the compactness theorem for FOL.

- Quasivariety: Class defined by quasiequations, or, equivalently, closed under isomorphisms,subalgebras, direct products, and ultraproducts.
- If Q is a quasivariety and $\mathbf{A} \in \mathsf{Q}$, a congruence Θ of $\mathbf{A}$ is a Q-congruence if $\mathbf{A}/\Theta \in \mathsf{Q}$.
- $\mathbf{A}$ is finitely Q-subdirectly irreducible if the least congruence Δ is meet-irreducible in $\mathrm{Con}_{\mathsf{Q}}(\mathbf{A})$.
- If Q is clear, we just call these relatively finitely subdirectly irreducible and denote the class of them by $\mathsf{Q}_{\mathsf{RFSI}}$.
- Q is Q-congruence distributive if $\mathrm{Con}_{\mathsf{Q}}(\mathbf{A})$ is distributive.
- Q has the Q-congruence extension property for each $\mathbf{B} \in \mathsf{Q}$, if $\mathbf{A} \leq \mathbf{B}$ and $\Theta \in \mathrm{Con}_{\mathsf{Q}}(\mathbf{A})$, then there exists $\Psi \in \mathrm{Con}_{\mathsf{Q}}(\mathbf{B})$ such that $\Theta = \Psi \cap A^2$.

Theorem (F.-Metcalfe 2024):

Let K be a subclass of a quasivariety Q satisfying

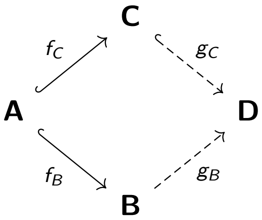
1. K is closed under isomorphisms and subalgebras;
2. every relatively subdirectly irreducible member of Q belongs to K;
3. for any $\mathbf{B} \in Q$ and subalgebra $\mathbf{A}$ of $\mathbf{B}$, if $\Theta \in \mathrm{Con}_{Q}\,\mathbf{A}$ and $\mathbf{A}/\Theta \in K$, then there exists a $\Phi \in \mathrm{Con}_{Q}\,\mathbf{B}$ such that $\Phi \cap A^2 = \Theta$ and $\mathbf{B}/\Phi \in K$;
4. every doubly injective span of finitely generated algebras in K has an amalgam in Q.

Then Q has the amalgamation property.

One-Sided Amalgamation

Definition:

Let K be a class of algebraic structures. We say that K has the one-sided amalgamation property (or 1AP) if for every span (A, B, C, f_B, f_C) in K there exists $D \in$ K, an embedding $g_B \colon B \to D$, and a homomorphism $g_C \colon C \to D$ such that $g_B \circ f_B = g_C \circ f_C$.



Characterizing the AP

Theorem (F.-Metcalfe 2023):

Let Q be any quasivariety with the Q-congruence extension property such that $\mathsf{Q}_{\mathrm{RFSI}}$ is closed under subalgebras. The following are equivalent:

1. Q has the amalgamation property.
2. Q has the one-sided amalgamation property.
3. $\mathsf{Q}_{\mathrm{RFSI}}$ has the one-sided amalgamation property.
4. Every doubly injective span in $\mathsf{Q}_{\mathrm{RFSI}}$ has an amalgam in $\mathsf{Q}_{\mathrm{RFSI}} \times \mathsf{Q}_{\mathrm{RFSI}}$.
5. Every doubly injective span of finitely generated algebras in $\mathsf{Q}_{\mathrm{RFSI}}$ has an amalgam in Q.

Decidability

Definition:

A variety finitely generated if it is generated as a variety by some given finite set of finite algebras of finite signature.

Theorem (F.-Metcalfe 2024):

Let V be a finitely generated congruence-distributive variety such that V_{FSI} is closed under subalgebras. There exist effective algorithms to decide if V has the congruence extension property and amalgamation property.

Essential extensions

There are some other formulations of the amalgamation property that are especially useful for studying algebraic structures whose FSIs have a particular form. Injective objects often play an important role, but we mention only the following, which is designed for totally ordered algebras.

Definition:

Let $\mathbf{A}, \mathbf{B}$ be algebras. An injective homomorphism $\varphi\colon \mathbf{A} \to \mathbf{B}$ is called an essential embedding if every congruence Θ of $\mathbf{B}$, $\Theta \neq \Delta_B$ implies $\Theta \cap \varphi[A]^2 \neq \Delta_{\varphi[A]}$. A span $\langle f_B\colon \mathbf{A} \to \mathbf{B}, f_C\colon \mathbf{A} \to \mathbf{C}\rangle$ is essential if f_C is essential, and a class K has the essential AP if every essential span in K has an amalgam in K.

Theorem (F.-Santschi 2024+):

Let V be a variety with the CEP such that $\mathbb{HS}(\mathsf{V}_{\mathrm{FSI}}) = \mathsf{V}_{\mathrm{FSI}}$. Then V has the AP if and only if $\mathsf{V}_{\mathrm{FSI}}$ has the essential AP.

Part II:
What We Know

Some Milestones

- Interpolation for propositional and first-order classical logic (Craig 1957).
- Exactly 8 varieties of Heyting algebras with Craig/deductive interpolation (Maksimova 1977); uniform interpolation for all of these (Pitts 1992, Ghilardi and Zawadowski 2002).
- Failure of amalgamation for many algebras associated to relevance logics (Urquhart 1993); classification of DIP + CIP for relevance logics with mingle (Marchioni and Metcalfe 2012).
- Classification of varieties of MV-algerbas (essentially intervals in abelian lattice-ordered groups) with the AP (Di Nola and Lettieri 2000).
- Uncountably many extensions of Hájek's basic fuzzy logic without DIP (Montagna 2006); lots of positive partial results (Aguzzoli and Bianchi 2021, 2023).

A residuated lattice is an algebraic structure of the form $(A, \wedge, \vee, \cdot, \backslash, /, e)$ where

- $(A, \wedge, \vee)$ is a lattice,
- $(A, \cdot, e)$ is a monoid, and
- for all $x, y, z \in A$,

$$x \cdot y \leq z \iff y \leq x \backslash z \iff x \leq z / y.$$

We use all the expected terminology: Commutative, idempotent, totally ordered, linear, etc.

Semilinear: Subalgebra of a direct product of totally ordered residuated lattices.

Pointed: Additional constant f (usually called FL-algebras).

Basic Structural Rules

$$\frac{\Gamma_1, \Pi_1, \Pi_2, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \Pi_2, \Pi_1, \Gamma_2 \Rightarrow \Delta}\ (e) \qquad \frac{\Gamma_1, \Pi, \Pi, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \Pi, \Gamma_2 \Rightarrow \Delta}\ (c)$$

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \Pi, \Gamma_2 \Rightarrow \Delta}\ (i) \qquad \frac{\Pi \Rightarrow}{\Gamma_1, \Pi, \Gamma_2 \Rightarrow \Delta}\ (o)$$

$$\frac{\Gamma_1, \Pi_1, \Gamma_2 \Rightarrow \Delta \quad \Gamma_1, \Pi_2, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \Pi_1, \Pi_2, \Gamma_2 \Rightarrow \Delta}\ (m)$$

Basic Structural Rules: Commutativity's Critical Role

State of the art up until a few years ago:

- Lots of success finding commutative varieties with the AP, but little systemic understanding.
- Much worse without commutativity: Previously thought that there may be non-commutative varieties of RL with AP (Gil Férez-Ledda-Tsinakis 2015).
- Example given in 2020 by Gil-Férez, Jipsen, Metcalfe.
- Several natural examples involving the law of excluded middle (F.-Galatos 2024).

Commutative varieties/logics

Theorem (F.-Santschi 2024+):

- Each of the varieties of commutative residuated lattices and pointed commutative residuated lattices has continuum-many subvarieties with the AP.
- Consequently, each of FL_e and its falsum-free fragment have continuum-many axiomatic extensions with the DIP.

Theorem (F.-Santschi 2024+):

- Each of the varieties of commutative residuated lattices and pointed commutative residuated lattices has continuum-many subvarieties without the AP.
- Consequently, each of FL_e and its falsum-free fragment have continuum-many axiomatic extensions without the DIP, hence without CIP.

Linear Logic

- Actually, the previous results can be extended by adding an involution, bounds, and the linear logic modalities (! and ?). Works for both classical and intuitionistic linear logic.
- Here for the cases with DIP, we also get a restricted form of Craig interpolation:

$$!A \to !B \Rightarrow \exists I(\mathrm{var}(I) \subseteq \mathrm{var}(A) \cap \mathrm{var}(B), !A \to !I, !I \to !B)$$

- Proof relies on deep properties of abelian groups plus some constructions from algebraic linear logic.

Non-commutative varieties

Theorem (F.-Metcalfe-Santschi 2024+):

1. There are continuum-many varieties of idempotent semilinear residuated lattices that have the amalgamation property and contain non-commutative members.
2. There are continuum-many axiomatic extensions of $\mathsf{SemRL_{cm}}$ that have the deductive interpolation property in which the exchange rule is not derivable.

Theorem (F.-Metcalfe-Santschi 2024+):

1. There are exactly sixty varieties of commutative idempotent semilinear residuated lattices that have the amalgamation property.
2. There are exactly sixty axiomatic extensions of $\mathsf{SemRL_{ecm}}$ that have the deductive interpolation property.

More on non-commutative varieties

- The previous results concern residuated lattices rather than FL-algebras, and you can add the point f (modeling negation).
- In the case with f, we can still show that there are finitely many varieties with amalgamation, but there are many more ($> 12,500,000$).
- The proof uses totally different methods than the commutative case; it depends on Galatos's 2004 work on encoding the dynamics of bi-infinite words into residuated lattices.

Hájek's basic logic

- Hájek's basic logic BL is the logic of continuous t-norms (associative, commutative binary operations $*$ on $[0,1]$ such that 1 is a unit). One of the most basic fuzzy logics.
- Algebraic models are BL-algebras: Commutative, integral, pointed residuated lattices with $x(x \to y) = x \wedge y$ and $(x \to y) \vee (y \to x) = 1$. These are semilinear.
- Quite a history of AP in BL-algebras/DIP in BL:
  - Montagna (2006) proves AP for variety of all BL-algebras along with many natural subvarieties, but gives uncountably many subvarieties without AP.
  - Subsequent partial classifications by Aguzzoli and Bianchi (2021 and 2023).
  - Main open question: Complete classification. How many varieties of BL-algebras with AP are there (countable or uncountable)?

Amalgamation BL-algebras

Because BL-algebras are semilinear, the essential AP turns out to be a key tool for getting a classification. Actually, it is easier to do this for basic hoops (0-free subreducts of BL-algebras) and then lift it to BL-algebras. We get:

Theorem (F.-Santschi 2024+):

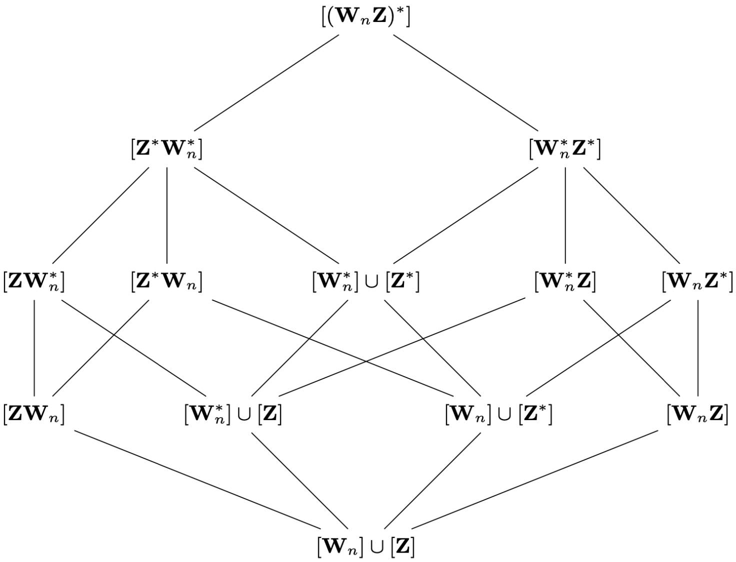
Let $\mathcal{BH}_{AP}$ be the poset of varieties of basic hoops with the AP. Then $(\mathcal{BH}_{AP}, \subseteq)$ can be partitioned into countably infinitely many finite intervals. In particular, there are only countably many varieties of basic hoops with the AP. A similar result holds for varieties of BL-algebras.

In particular, we can solve the open problem of Montagna:

Theorem (F.-Santschi 2024+):

Hájek's basic logic has only countably many axiomatic extensions with the DIP. The same holds for its negation-free fragment.

Example of an interval of $\mathcal{BH}_{AP}$



Heuristic methods can also be used to find lots of failures of AP, hence DIP (and often CIP). Eg:

Proposition (F. 2024+):

The variety of semilinear De Morgan monoids does not have the AP, and hence the semilinear extension of the basic relevant logic **R** does not have DIP. However, there are infinitely many axiomatic extension of **R** plus semilinearity with DIP.

A knotted rule is an equation of the form $x^n \leq x^m$, where $n \neq 0$.

Proposition (F.-Santschi 2024+):

The subvariety of residuated lattices axiomatized by (semilinearity + any knotted rule) does not have the AP. The same holds in the presence of commutativity, and in that case the result implies the failure of DIP and CIP for the corresponding logics.

Summing up

- We now have a very thorough picture of DIP for most of the logics defined by the basic structural rules.
- Essentially complete understanding of AP for the prominent varieties of semilinear residuated lattices.
- The toolkit has allowed us to solve a lot of the longstanding open problems, e.g. for logics with exchange and for Hájek's basic fuzzy logic.
- Not yet automatic, but some progress on concrete algorithms.
- Heuristic tools are very effective for many purposes (often small counterexamples).

Part III:
What We'd Like to Know

Charting the Basic Structural Rules

Question:

How many varieties of idempotent commutative residuated lattices (no semilinearity) have AP?

Question:

Are there continuum-many varieties of RL's with $x \leq 1$ that have AP? What about if we add commutativity?

Computation and finiteness

Question:

Let V be a congruence distributive variety whose class of finitely subdirectly irreducibles is closed under subalgebras. If V is finitely presented, is it decidable whether V has CEP, AP, and so forth?

Extending this body of work to finitely presented algebras is especially interesting because of its connection to uniform interpolation: A variety V has right uniform deductive interpolation if and only it has DIP and is coherent (any finitely generated subalgebra of a finitely presented algebra is finitely presented).

New Transfer Theorems

Question:

Can we develop good transfer theorems for studying Craig interpolation, uniform interpolation, and so on?

Question:

Can we obtain more convenient transfer theorems for studying quasivarieties, in particular in terms of critical algebras?

Jónsson's Problem

Question (Jónsson's Problem):

Are there techniques for determining whether a class of mathematical structures has AP based solely on the 'shape' of the defining axioms? Can we at least do so for varieties or quasivarieties of (congruence distributive) algebras? Or for residuated lattices?

Thank you!