Interpolation in basic fuzzy logics

Wesley Fussner

Institute of Computer Science
Czech Academy of Sciences

Seminar of the Department of Logic,
Charles University, Prague, Czechia

13 May 2024

Part I: Preamble

The idea of interpolation

Interpolation refers to a cluster of metalogical properties asserting that if A entails B, then there exists I dealing only with the subject matter common to A and B and such that

$$A \text{ entails } I \text{ and } I \text{ entails } B.$$

I is called an interpolant and gives a kind of explanation for why A entails B.

The intuitive notions 'entailment' and 'common subject matter' are cashed out in many different ways.

Interpolation: Some Variants

Craig interpolation:

$$A \to B \Rightarrow \exists I(\text{var}(I) \subseteq \text{var}(A) \cap \text{var}(B), A \to I, I \to B)$$

Deductive interpolation:

$$A \vdash B \Rightarrow \exists I(\text{var}(I) \subseteq \text{var}(A) \cap \text{var}(B), A \vdash I, I \vdash B)$$

Uniform interpolation: For any $S \subseteq \text{var}(A)$, there exists I, $\text{var}(I) \subseteq S$, so that if $\text{var}(A) \cap \text{var}(B) \subseteq S$,

$$A \to B \iff I \to B$$

Interpolation: Applications

- Philosophical issues (e.g. argumentation theory)
- Hardware and software verification (Ken McMillan approach)
- Database theory (primarily uniform interpolation)
- Recent efforts to use interpolants as a resource for SMT solving.

Toward a coherent picture

Interpolation turned out to be a treacherous subject. There are many published proofs by well known researchers that were later found to be wrong, leaving the result open, with neither proofs nor counter-examples. Although in general we have some ideas of how and why interpolation may hold or fail for a given system, we still have the persistent feeling that really we need to obtain our results (if we can) logic by logic, case by case, and that a slight variation in the logic may change the outcome. Yet we feel that interpolation is a coherent research area and we just have to wait and see to figure out what is going on.

-D. Gabbay and L. Maksimova,
Interpolation and Definability.
Oxford University Press, 2005.

Interpolation: A Short Bibliography

This represents a case study, as part of a larger effort to see 'what is going on':

- W. Fussner and N. Galatos, Semiconic idempotent logic I: Structure and local deduction theorems. Ann. Pure Appl. Logic 175, 103443 (2024). `https://doi.org/10.1016/j.apal.2024.103443`
- W. Fussner and N. Galatos, Semiconic idempotent logic II: Beth definability and deductive interpolation. To appear in Ann. Pure Appl. Logic. `https://arxiv.org/abs/2208.09724v3`.
- W. Fussner and G. Metcalfe, Transfer theorems for finitely subdirectly irreducible algebras. J. Algebra 640, pp. 1-20 (2024). `https://doi.org/10.1016/j.jalgebra.2023.11.003`.

Interpolation: A Short Bibliography

- W. Fussner, G. Metcalfe, and S. Santschi, Interpolation and the Exchange rule. https://arxiv.org/abs/2310.14953
- W. Fussner and S. Santschi, Interpolation in Linear Logic and Related Systems. https://arxiv.org/abs/2305.05051
- W. Fussner and S. Santschi, Interpolation in Hájek's Basic Logic. https://arxiv.org/abs/2403.13617
- W. Fussner, Revisiting Interpolation in Relevant Logics. Manuscript.
- W. Fussner and S. Santschi, Amalgamation in Semilinear Residuated Lattices. Manuscript.

Triangular norms and their logic

Petr Hájek's basic logic is the logic of continuous triangular norms.

A triangular norm (or t-norm) is a commutative and associative binary operation $\cdot$ on $[0, 1]$ that has 1 as an identity element.

Every continuous t-norm $\cdot$ has an associated residual operation, i.e binary operations $\to$ on $[0, 1]$ such that

$$x \cdot y \leq z \iff x \leq y \to z.$$

Binary infimum $\wedge$ and supremum $\vee$ are definable from $\cdot$ and $\to$ alone:

$$x \wedge y := x \cdot (x \to y),$$

$$x \vee y := ((x \to y) \to y) \wedge ((y \to x) \to x).$$

Hájek's basic fuzzy logic consists of exactly the formulas φ in the language $\wedge, \vee, \cdot, \to, 0, 1$ such that $\varphi = 1$ holds for any continuous t-norm algebra on $[0, 1]$. Hájek axiomatized this logic in his 1999 monograph, and Karel Chvalovský later showed that the axiomatization could be given in the following simplified format. Here & is interpreted by $\cdot$, and the only rule is modus ponens.

$$
\begin{aligned}
&\text{(A1)} \quad (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)),\\
&\text{(A2)} \quad (\varphi \,\&\, \psi) \to \varphi,\\
&\text{(A3)} \quad (\varphi \,\&\, \psi) \to (\psi \,\&\, \varphi),\\
&\text{(A4)} \quad (\varphi \,\&\, (\varphi \to \psi)) \to (\psi \,\&\, (\psi \to \varphi)),\\
&\text{(A5a)} \quad (\varphi \to (\psi \to \chi)) \to ((\varphi \,\&\, \psi) \to \chi),\\
&\text{(A5b)} \quad ((\varphi \,\&\, \psi) \to \chi) \to (\varphi \to (\psi \to \chi)),\\
&\text{(A6)} \quad ((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi),\\
&\text{(A7)} \quad \bot \to \varphi.
\end{aligned}
$$

BL-algebras

Although they are the basic models, t-norm based algebras are not the only algebraic models of *BL*:

Definition:

A BL-algebra is an algebra $(A, \wedge, \vee, \cdot, \rightarrow, 0, 1)$ such that $(A, \wedge, \vee)$ is a lattice with bounds 0 and 1, $(A, \cdot, 1)$ is a monoid, $\rightarrow$ is the residual of $\cdot$, and

$$x \cdot (x \rightarrow y) = x \wedge y \text{ and } (x \rightarrow y) \vee (y \rightarrow x) = 1.$$

BL-algebras form a variety and one may show that this variety is semilinear (i.e., generated by totally ordered algebras).

Part II:
Our Toolkit

Interpolation via Amalgamation

Interpolation has been widely studied with proof theory, but proof-theoretic methods turn out to be unsuitable for studying extensions of BL. For this, we approach interpolation via amalgamation:

Theorem (Czelakowski-Pigozzi 1999):

Let $\vdash$ be an algebraizable deductive system with a local deduction theorem whose equivalent algebraic semantics is the variety $\mathcal{V}$. Then $\vdash$ has the deductive interpolation property if and only if $\mathcal{V}$ has the amalgamation property.

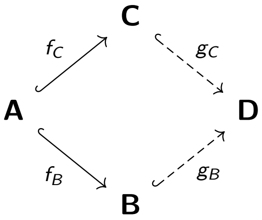
Major question: How to establish/refute the AP.

Answer: Reduce the complexity of the problem to some tractable generating class.

Amalgamation

Definition:

Let K be a class of algebraic structures. A span in K is a quintuple (A, B, C, f_B, f_C), where $A, B, C \in$ K and $f_B \colon A \to B$, $f_C \colon A \to C$ are embeddings. We say that K has the amalgamation property (or AP) if for every span (A, B, C, f_B, f_C) in K there exists $D \in$ K and embeddings $g_B \colon B \to D$ and $g_C \colon C \to D$ such that
$g_B \circ f_B = g_C \circ f_C$.



The idea of amalgamation

Some familiar groups:

*	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

*	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

The idea of amalgamation

First attempt to embed both groups in a bigger group:

*	0	1	2	a	b	c
0	0	1	2	a	b	c
1	1	2	0			
2	2	0	1			
a	a			0	c	b
b	b			c	0	a
c	c			b	a	0

Turns out to be impossible on 6 elements...

BUT you can embed both groups in their direct product.

Finitely generated algebras are enough

Theorem (F.-Metcalfe 2024)

Let K be a universal class of algebras such that every span of finitely generated algebras in K has an amalgam in K. Then K has the amalgamation property.

Proof: Let (A, B, C, f_B, f_C) be a span in K, assuming WLOG that f_B and f_C are just inclusion maps.

Let Σ be the union of the theory of K with the atomic diagrams of B and C. Then the span has an amalgam iff Σ has a model.

Now consider Σ' to be the theory of K with any finite subset of the atomic diagrams of B and C, giving respective subalgebras A', B', C' generated by the finitely many elements named in Σ'.

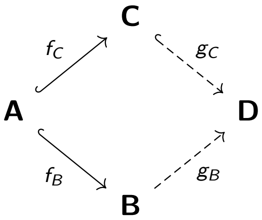
By assumption (A', B', C', f'_B, f'_C) has an amalgam in K. So Σ has a model by the compactness theorem for FOL.

- Quasivariety: Class defined by quasiequations, or, equivalently, closed under isomorphisms,subalgebras, direct products, and ultraproducts.
- If Q is a quasivariety and $\mathbf{A} \in \mathsf{Q}$, a congruence Θ of $\mathbf{A}$ is a Q-congruence if $\mathbf{A}/\Theta \in \mathsf{Q}$.
- $\mathbf{A}$ is finitely Q-subdirectly irreducible if the least congruence Δ is meet-irreducible in $\mathrm{Con}_{\mathsf{Q}}(\mathbf{A})$.
- If Q is clear, we just call these relatively finitely subdirectly irreducible and denote the class of them by $\mathsf{Q}_{\mathsf{RFSI}}$.
- Q is Q-congruence distributive if $\mathrm{Con}_{\mathsf{Q}}(\mathbf{A})$ is distributive.
- Q has the Q-congruence extension property for each $\mathbf{B} \in \mathsf{Q}$, if $\mathbf{A} \leq \mathbf{B}$ and $\Theta \in \mathrm{Con}_{\mathsf{Q}}(\mathbf{A})$, then there exists $\Psi \in \mathrm{Con}_{\mathsf{Q}}(\mathbf{B})$ such that $\Theta = \Psi \cap A^2$.

One-Sided Amalgamation

Definition:

Let K be a class of algebraic structures. We say that K has the one-sided amalgamation property (or 1AP) if for every span (A, B, C, f_B, f_C) in K there exists $D \in$ K, an embedding $g_B \colon B \to D$, and a homomorphism $g_C \colon C \to D$ such that $g_B \circ f_B = g_C \circ f_C$.



Characterizing the AP

Theorem (F.-Metcalfe 2023):

Let Q be any quasivariety with the Q-congruence extension property such that Q_{RFSI} is closed under subalgebras. The following are equivalent:

1. Q has the amalgamation property.
2. Q has the one-sided amalgamation property.
3. Q_{RFSI} has the one-sided amalgamation property.

Essential extensions

The following formulation of amalgamation is designed specifically for varieties of BL-algebras.

Definition:

Let $\mathbf{A}, \mathbf{B}$ be algebras. An injective homomorphism $\varphi\colon \mathbf{A} \to \mathbf{B}$ is called an essential embedding if every congruence Θ of $\mathbf{B}$, $\Theta \neq \Delta_B$ implies $\Theta \cap \varphi[A]^2 \neq \Delta_{\varphi[A]}$. A span $\langle f_B\colon \mathbf{A} \to \mathbf{B}, f_C\colon \mathbf{A} \to \mathbf{C}\rangle$ is essential if f_C is essential, and a class K has the essential AP if every essential span in K has an amalgam in K.

Theorem (F.-Santschi 2024+):

Let V be a variety with the CEP such that $\mathbb{HS}(\mathsf{V}_{\mathsf{FSI}}) = \mathsf{V}_{\mathsf{FSI}}$. Then V has the AP if and only if $\mathsf{V}_{\mathsf{FSI}}$ has the essential AP.

Part III:
Deductive Interpolation in BL

Quite a history of AP in BL-algebras/DIP in BL:

- Montagna (2006) proves AP for variety of all BL-algebras along with many natural subvarieties, but gives uncountably many subvarieties without AP.
- Subsequent partial classifications by Aguzzoli and Bianchi (2021 and 2023).
- Main open question: Complete classification. How many varieties of BL-algebras with AP are there (countable or uncountable)?

BL-algebras: Some structure theory

We will classify the extensions of BL with deductive interpolation by classifying varieties of BL-algebras with the amalgamation property. For this, we will need to understand the structure of finitely subdirectly irreducible BL-algebras, AKA totally ordered BL-algebras.

Theorem (Agliano-Montagna 2003):

Every totally ordered BL-algebra may be represented uniquely as an ordinal sum of Wajsberg hoops.

For an abelian lattice-orderd group $\mathbf{G}$ with an element $u \geq 0$, we can define the MV-algebra $\mathbf{\Gamma}(\mathbf{G}, u) = ([0, u], \wedge, \vee, \cdot, \rightarrow, u, 0)$ as follows:

- $[0, u] = \{x \in G \mid 0 \leq x \leq u\}$,
- the meet and join are just the restriction of the meet and join of $\mathbf{G}$ to $[0, u]$,
- for $a, b \in [0, u]$, $a \cdot b = (a + b - u) \vee 0$,
- for $a, b \in [0, u]$, $a \rightarrow b = (u + b - a) \wedge u$.

BL-algebras: Some building blocks

We will construct all of the varieties with the AP by considering these basic building blocks, where min $\mathbb{N}$:

- $\mathbf{Ł}_m = \mathbf{\Gamma}(\mathbb{Z}, m)$
- $\mathbf{Ł}_{m,\omega} = \mathbf{\Gamma}(\mathbb{Z} \times \mathbb{Z}, (m, 0))$ (here $\mathbb{Z} \times \mathbb{Z}$ is lexicographically ordered from the left)
- $\mathbf{W}_m$ and $\mathbf{W}_{m,\omega}$ the 0-free reducts of $\mathbf{Ł}_m$ and $\mathbf{Ł}_{m,\omega}$, respectively.
- $\mathbf{Z}$ denotes the negative integers (as a lattice-ordered group).

The algebras $\mathbf{Ł}_m$ and $\mathbf{Ł}_{m,\omega}$ are MV-algebras, whereas the others are Wajsberg hoops. There are complete classifications of varieties of MV-algebras/Wajsberg hoops with the AP.

BL-algebras: Some notation

It turns out that naming the varieties of BL-algebras with amalgamation is pretty non-trivial. For this, we resort to using some regular expressions.

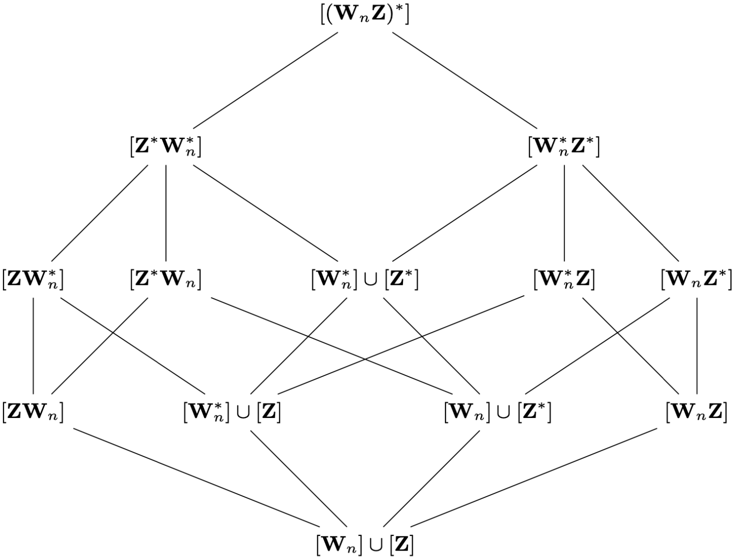
- We write the ordinal sum $\mathbf{A} \oplus \mathbf{B}$ by $\mathbf{AB}$.
- $[\mathbf{AB}]$ denotes the variety generated by all ordinal sums of the form $\mathbf{A}' \oplus \mathbf{B}'$ where $\mathbf{A}' \in \mathrm{HSP_u}(\mathbf{A})$ and $\mathbf{B}' \in \mathrm{HSP_u}(\mathbf{B})$.
- We use Kleene star * to denote the repetition of one or more instances of a summand in an ordinal sum, e.g., $[\mathbf{AB}^*]$ abbreviates the class consisting of all ordinal sums of the form $\mathbf{A} \oplus \mathbf{B}_1 \oplus \cdots \oplus \mathbf{B}_n$, where n is a positive integer and $\mathbf{B}_1, \ldots, \mathbf{B}_n \in \mathrm{HSP_u}(\mathbf{B})$.

Amalgamation basic hoops

Because BL-algebras are semilinear, the essential AP turns out to be a key tool for getting a classification. Actually, it is easier to do this for basic hoops (0-free subreducts of BL-algebras) and then lift it to BL-algebras. We get:

Theorem (F.-Santschi 2024+):

Let $\mathcal{BH}_{AP}$ be the poset of varieties of basic hoops with the AP. Then $(\mathcal{BH}_{AP}, \subseteq)$ can be partitioned into countably infinitely many finite intervals. In particular, there are only countably many varieties of basic hoops with the AP. A similar result holds for varieties of BL-algebras.


$[(\mathbf{W}_n\mathbf{Z})^*]$
$[\mathbf{Z}^*\mathbf{W}_n^*]$
$[\mathbf{W}_n^*\mathbf{Z}^*]$
$[\mathbf{Z}\mathbf{W}_n^*]$
$[\mathbf{Z}^*\mathbf{W}_n]$
$[\mathbf{W}_n^*] \cup [\mathbf{Z}^*]$
$[\mathbf{W}_n^*\mathbf{Z}]$
$[\mathbf{W}_n\mathbf{Z}^*]$
$[\mathbf{Z}\mathbf{W}_n]$
$[\mathbf{W}_n^*] \cup [\mathbf{Z}]$
$[\mathbf{W}_n] \cup [\mathbf{Z}^*]$
$[\mathbf{W}_n\mathbf{Z}]$
$[\mathbf{W}_n] \cup [\mathbf{Z}]$

Lifting the result to BL-algebras

A similar description can be given for varieties of BL-algebras with the amalgamation property. Using these results, plus the fact that *BL* has BL-algebras as its equivalent algebraic semantics, we may solve Montagna's problem:

Theorem (F.-Santschi 2024+):

Hájek's basic logic has only countably many axiomatic extensions with the deductive interpolation property. The same holds for its negation-free fragment.

Montgana showed that the only extensions of *BL* with Craig interpolation are the extensions that are also superintuitionistic logics (already classified by Maksimova), so this completes the description of axiomatic extensions of *BL* with both Craig and deductive interpolation.

A sketch of the proof

The proof contains many moving components, but overall it proceeds as follows.

- First, since every variety is generated by its finitely generated algebras, it is enough to consider varieties of basic hoops that are generated by ordinal sums with finitely many summands in their decomposition.
- Now, once the generating algebras (appearing in the nodes of the finite intervals we have exhibited) are identified, the proof amounts to
  1. showing that the generating classes for each of the generated varieties has the essential AP; and
  2. showing that these are all varieties with the AP using closure conditions.

A sketch of the proof

These closure conditions assert that if a variety V has the AP and contains a certain algebra, then it must also contain some other identified algebra. For example:

Lemma:

Let V be a variety of basic hoops with the amalgamation property. Let V_{fc} be the class of its members that may be written as ordinal sums with finitely many summands, and $\mathrm{Wajs}(\mathsf{V})$ be the class of Wajsberg chains appearing as summands in V. If $\bigoplus_{i=1}^{l} \mathbf{A}_i \in \mathsf{V}_{\mathrm{fc}}$, $\mathbf{B} \in \mathrm{Wajs}(\mathsf{V})$ is simple, and for $1 \leq k \leq l$, $\mathbf{A}_k \in [\mathbf{B}]$ is non-trivial, then $(\bigoplus_{i=1}^{k-1} \mathbf{A}_i) \oplus \mathbf{B} \oplus (\bigoplus_{i=k+1}^{l} \mathbf{A}_i) \in \mathsf{V}_{\mathrm{fc}}$.

Summing up

- We now have a very thorough picture of deductive interpolation for most of the logics defined by the basic structural rules.
- Essentially complete understanding for logics characterized by linearly ordered algebraic models, with *BL* having been one of the last prominent outliers.
- Tools that worked for the *BL* case may also be adapted to other logics where models have nice ordinal sum decompositions (e.g., some logics of lower-semicontinuous t-norms).

Thank you!