Amalgamation in Varieties of BL-algebras

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Hájek's basic fuzzy logic and BL-algebras

- Hájek's basic logic is the logic of continuous t-norms on $[0, 1]$, and its equivalent algebraic semantics is the variety of BL-algebras.
- Extensively studied: free algebras, very nice structure theory, decidability and complexity results, relational semantics, etc.
- Persistent open problem: Classification of subvarieties of **BL** is the amalgamation property.
- AP corresponds to the deductive interpolation property for the associated extensions of basic logic (we already have a classification for CIP).

Amalgamation in BL-algebras: a short history

Quite a history of trying to pin down a classification of subvarieties with the AP:

- Montagna 2006: Variety of all BL-algebras + many of the most natural subvarieties have AP, but there are uncountably many that do not.
- Montagna's problem: How many varieties of BL-algebras have AP? Countably many or uncountably many?
- Cortonesi, Marchioni, and Montagna 2011: Applied tools from first-order model theory.
- Aguzzoli and Bianchi 2021: Partial classification for finitely generated varieties.
- Fussner and Metcalfe 2022: New general results for studying AP.
- Aguzzoli and Bianchi 2023: Sharpened classification, but still not complete.

The main theorem

Let $\mathcal{BL}_{\mathrm{AP}}$ be the subposet of the subvariety lattice of BL-algebras consisting of the subvarieties with the AP.

Theorem:

$\mathcal{BL}_{\mathrm{AP}}$ is the union of countably infinitely many finite intervals, which may be described explicitly.

Actually, the proof passes through basic hoops (0-free subreducts of BL-algebras) and we get a similar theorem for them. This involves fewer technicalities so we will focus on the basic hoop case.

The proof has a lot of moving pieces, so we will give the basic definitions and focus on the two key ideas that allowed us to succeed: the essential amalgamation property and an auspicious nomenclature for varieties in terms of regular expressions.

A class K of algebras has the amalgamation property if every span $\langle \phi_1\colon \mathbf{A} \to \mathbf{B}, \phi_2\colon \mathbf{A} \to \mathbf{C} \rangle$ of algebras in K can be completed in K:

$$\begin{tikzcd}
 & \mathbf{B} \\
\mathbf{A} \arrow[ur, hook, "\phi_1"] \arrow[dr, hook, "\phi_2"'] & \\
 & \mathbf{C}
\end{tikzcd}$$

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$$\begin{tikzcd}
& \mathbf{B} \arrow[dr, hook, dashed, "\psi_1"] & \\
\mathbf{A} \arrow[ur, hook, "\phi_1"] \arrow[dr, hook, "\phi_2"'] & & \mathbf{D} \\
& \mathbf{C} \arrow[ur, hook, dashed, "\psi_2"'] &
\end{tikzcd}$$

An commutative integral residuated lattice is an algebraic structure of the form $(A, \wedge, \vee, \cdot, \rightarrow, 1)$ where

- $(A, \wedge, \vee, 1)$ is a lattice with top element 1,
- $(A, \cdot, 1)$ is a monoid, and
- for all $x, y, z \in A$,

$$x \cdot y \leq z \iff x \leq y \rightarrow z.$$

A BL-algebras is a pointed commutative integral residuated lattice (with additional constant 0 for the bottom element) satisfying $(x \rightarrow y) \vee (y \rightarrow x) \approx 1$ and $x \cdot (x \rightarrow y) \approx x \wedge y$.

A basic hoop is a 0-free subreduct of a BL-algebra.

Key idea 1: essential extensions

There are two basic ingredients that allow us to get the classification. The first of these is a new presentation of AP.

An extension $\mathbf{A} \leq \mathbf{B}$ is essential if for every congruence $\theta \neq \Delta_{\mathbf{B}}$ of $\mathbf{B}$, $\theta \cap A^2 \neq \Delta_{\mathbf{A}}$.

An embedding $\varphi\colon \mathbf{A} \to \mathbf{B}$ is essential if $\varphi[\mathbf{A}] \leq \mathbf{B}$ is.

A span $\langle f_B\colon \mathbf{A} \to \mathbf{B}, f_C\colon \mathbf{A} \to \mathbf{C}\rangle$ is essential if f_C is essential, and a class K has the essential AP if every essential span in K has an amalgam in K.

Theorem:

Let V be a variety with the CEP such that $\mathbb{HS}(\mathsf{V}_{\mathsf{FSI}}) = \mathsf{V}_{\mathsf{FSI}}$. Then V has the AP if and only if $\mathsf{V}_{\mathsf{FSI}}$ has the essential AP.

Key idea 1: essential closedness

Let $\mathsf{K}_1, \mathsf{K}_2$ be classes of similar algebras with $\mathsf{K}_1 \subseteq \mathsf{K}_2$ We say that K_1 is essentially closed in K_2 if for any $\mathbf{A} \in \mathsf{K}_1$ and essential embedding $\varphi\colon \mathbf{A} \to \mathbf{B}$ with $\mathbf{B} \in \mathsf{K}_2$, we also have $\mathbf{B} \in \mathsf{K}_1$.

Lemma:

Let $\mathsf{K}_1, \mathsf{K}_2, \mathsf{K}_3$ be classes of similar algebras.

1. If K_1 is essentially closed in K_2 and K_3, then K_1 is essentially closed in $\mathsf{K}_2 \cup \mathsf{K}_3$.
2. If K_1 and K_2 are essentially closed in K_3, then $\mathsf{K}_1 \cap \mathsf{K}_2$ is essentially closed in K_3.
3. If K_1 is essentially closed in K_2 and K_2 is essentially closed in K_3, then K_1 is essentially closed in K_3.

Key idea 1: essential closedness and the essential AP

Lemma:

Suppose $\mathsf{K} = \mathsf{K}_1 \cup \mathsf{K}_2$, $\mathbb{IS}(\mathsf{K}_i) = \mathsf{K}_i$ for $i = 1, 2$, and $\mathsf{K}_1 \cap \mathsf{K}_2$ is essentially closed in K. Then:

1. If K_1 and K_2 have the essential amalgamation property, then K has the essential amalgamation property.
2. If K and $\mathsf{K}_1 \cap \mathsf{K}_2$ have the essential amalgamation property, then K_1 and K_2 have the essential amalgamation property.

Interlude: the structure of totally ordered basic hoops

BL-algebras and basic hoops are semilinear (their FSIs are exactly the totally ordered ones), so all of the previous tools are very well suited to studying them.

A Wajsberg hoop is an integral commutative residuated lattice satisfying $(x \to y) \to y \approx x \vee y$. Every Wajsberg hoop is either bounded (in which case it is a 0-free reduct of an MV-algebra) or it is cancellative.

Aglianò and Montagna (2003) showed that every basic hoop $\mathbf{A}$ is an ordinal sum $\mathbf{A} \cong \bigoplus_{i \in I} \mathbf{A}_i$ of totally ordered Wajsberg hoops. If I is finite, then $\mathbf{A}$ has finite index

Every variety of basic hoops is generated by its finite index members. It is enough to consider these to study the AP.

Denote by:

- $\mathbf{Z}$ the negative cone of the integers (a cancellative hoop)
- $\mathbf{W}_m$ the 0-free reduct of the MV-algebra $Ł_m$.
- $\mathbf{W}_{m,\omega}$ the 0-free reduct of the MV-algebra $\Gamma(\mathbb{Z} \times \mathbb{Z}, \langle m, 0 \rangle)$, where $\mathbb{Z} \times \mathbb{Z}$ is ordered lexicographically as an ℓ-group and Γ is the Mundici functor.

Note that the varieties of MV-algebras and Wajsberg hoops with the AP have already been classified (by Di Nola-Lettieri and Metcalfe-Montagna-Tsinakis).

Key idea 2: nomenclature

We introduce some naming conventions for varieties:

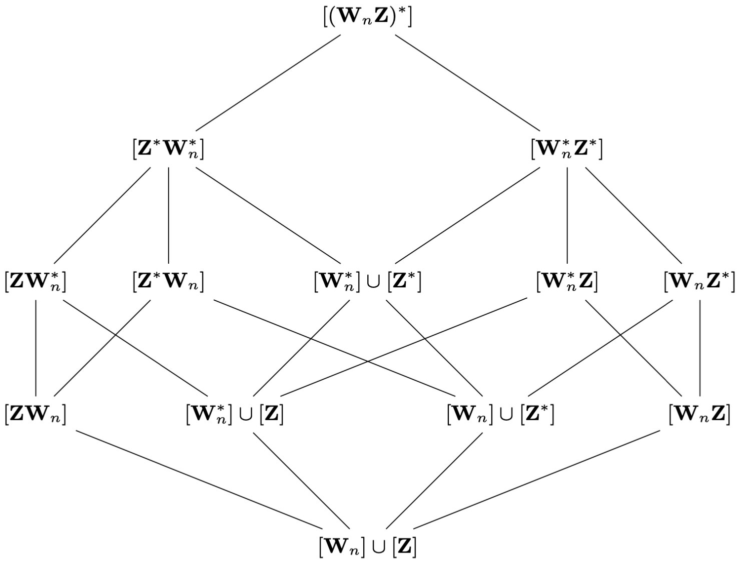
- $\mathbf{A} \oplus \mathbf{B}$ is written $\mathbf{AB}$
- The class generated by the componentwise $\mathrm{HSP_u}$ closure of an ordinal sum by enclosing the corresponding ordinal sum in bracket [,], so that, for example, $[\mathbf{AB}]$ denotes the class of all ordinal sums $\mathbf{A}' \oplus \mathbf{B}'$ where $\mathbf{A}' \in \mathrm{HSP_u}(\mathbf{A})$ and $\mathbf{B}' \in \mathrm{HSP_u}(\mathbf{B})$; and $[\mathbf{A}] = \mathrm{HSP_u}(\mathbf{A})$.
- We use * to denote the repetition of one or more instances of a summand in a given ordinal sum. For example, $[\mathbf{AB}^*]$ abbreviates the class consisting of all ordinal sums of the form $\mathbf{A} \oplus \mathbf{B}_1 \oplus \cdots \oplus \mathbf{B}_n$, where n is a positive integer and $\mathbf{B}_1, \ldots, \mathbf{B}_n \in \mathrm{HSP_u}(\mathbf{B})$.
- Kleene star * has priority over $\oplus$, so that $[\mathbf{ABC}^*]$ abbreviates $[(\mathbf{A} \oplus \mathbf{B}) \oplus \mathbf{C}^*]$.

The main theorem

Theorem:

The poset of varieties of basic hoops may be written as the union of countably infinitely many finite intervals, so in particular there are only countably many varieties of basic hoops with the AP. Similarly, $\mathcal{BL}_{\mathrm{AP}}$ is the union of countably infinitely many finite intervals.

Example of an interval of $\mathcal{BH}_{AP}$



Final remarks

- Essential AP and essential closedness are powerful tools that show promise for the study of AP in other varieties.
- In particular, the tools we use here will work with some modification to other varieties with good ordinal sum representations (e.g. this could be done for certain varieties of MTL-algebras).
- We still don't know if there are continuum-many varieties of $\mathbf{FL}_{ew}$-algebras with the AP.

For more information, see:

W. Fussner and S. Santschi, Interpolation in Hájek's Basic Logic, `https://arxiv.org/abs/2403.13617`.

Thank you!