Interpolation in Some Modal Substructural Logics

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Interpolation: a brief history

Various kinds of interpolation properties have been studied quite a lot:

- Just seven consistent superintuitionistic logics with CIP/DIP (Maksimova)
- At most 49 consistent normal extensions of **S4** with DIP (Maksimova)
- CIP for basic substructural logics like **FL**, $\mathbf{FL_e}$, $\mathbf{FL_c}$, etc
- Countably infinitely many extensions of Łukasiewicz's infintie valued logic with DIP (Di Nola–Lettieri)

Interpolation: main themes

Big question: How do we make sense of this zoo of different results?

Convention wisdom: Interpolation is a rather uncommon property

Medium question: Is this true? How common is interpolation?

Motivating question for today: Are there uncountably many extensions of $\mathbf{FL_e}$ with interpolation?

Main theorem:

There are continuum-many axiomatic extensions of $\mathbf{FL_e}$ with the deductive interpolation property, and this remains true for many extensions of the language: with involution, **S4**-like modals, and for full classical linear logic.

Interpolation: variants

Craig interpolation:

$$A \to B \Rightarrow \exists I(\mathrm{var}(I) \subseteq \mathrm{var}(A) \cap \mathrm{var}(B), A \to I, I \to B)$$

Deductive interpolation:

$$A \vdash B \Rightarrow \exists I(\mathrm{var}(I) \subseteq \mathrm{var}(A) \cap \mathrm{var}(B), A \vdash I, I \vdash B)$$

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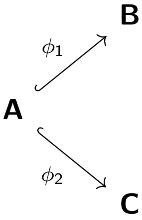
Interpolation done algebraically

Theorem (Czelakowski-Dziobiak):

Let $\vdash$ be a strongly algebraizable deductive system with a local deduction theorem and equivalent algebraic semantics V. Then $\vdash$ has the DIP if and only if V has the amalgamation property.



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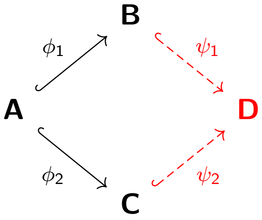


Residuated lattices: the basic signature

An commutative residuated lattice is an algebraic structure of the form $(A, \wedge, \vee, \cdot, \rightarrow, 1)$ where

- $(A, \wedge, \vee)$ is a lattice,
- $(A, \cdot, 1)$ is a monoid, and
- for all $x, y, z \in A$,

$$x \cdot y \leq z \iff x \leq y \rightarrow z.$$

Residuated lattices: expansions

An $\mathbf{FL_e}$-algebras has an additional constant 0, and it is involutive if $x = (x \to 0) \to 0$.

A girale is a bounded involutive $\mathbf{FL_e}$-algebra with an extra unary operation ! satisfying:

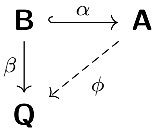
- $!(x \wedge y) = !x \cdot !y$,
- $!!x = !x \leq x \wedge 1$,
- $!1 = 1$.

Agliano (mid-1990s): Girales are the equivalent algebraic semantics of classical linear logic (? and $\invamp$ are definable).

The construction of the varieties with AP works for the basic residuated lattice signatures plus any combination of bounds, 0, involution, and the modal !. We will focus on girales.

Building the extensions: injectives

An algebra $\mathbf{Q}$ in a class K is called injective if for all $\mathbf{A}, \mathbf{B} \in$ K, every embedding $\alpha\colon \mathbf{B} \to \mathbf{A}$, and every homomorphism $\beta\colon \mathbf{B} \to \mathbf{Q}$, there exists $\varphi\colon \mathbf{A} \to \mathbf{Q}$ with $\varphi \circ \alpha = \beta$:



A class K has enough injectives if every algebra in K embeds into an algebra in K that is injective over K)

Lemma (folklore):

Suppose K is a class of similar algebras that is closed under finite products. If K has enough injectives, then K has the amalgamation property.

Building the extensions: starting from abelian groups

A subgroup $\mathbf{G}$ of a group $\mathbf{H}$ is an essential subgroup of $\mathbf{H}$ if for every non-trivial subgroup $\mathbf{G}'$ of $\mathbf{H}$ we have that $G \cap G'$ is non-trivial.

Lemma (Eckmann–Schopf 1953):

Every abelian group is an essential subgroup of an injective abelian group.

For each set P of primes numbers we define a set of quasiequations by

$$\Sigma_P = \{x^p \approx 1 \Rightarrow x \approx 1 \mid p \in P\}.$$

The quasivariety of abelian groups defined by Σ_P is called Q_P.

Building the extensions: starting from abelian groups

Lemma:

For every set of prime numbers P, the quasivariety Q_P has the amalgamation property.

Proof: It's enough to show that Q_P has enough injectives.

Let $\mathbf{G} \in \mathsf{Q}_P$. By the lemma, $\mathbf{G}$ is an essential subgroup of an injective abelian group $\mathbf{H}$. it suffices to show that $\mathbf{H} \in \mathsf{Q}_P$.

Toward a contradiction, suppose $\mathbf{H} \notin \mathsf{Q}_P$. Then there is $p \in P$ and $a \in H$ with $a^p = 1$ and $a \neq 1$.

The subgroup $\mathbf{S}$ generated by a is cyclic of order p and $\mathbf{G}$ is an essential subgroup of $\mathbf{H}$, so there exists $b \in S \cap G$ with $b \neq 1$. But then $b^p = 1$, so $\mathbf{G} \notin \mathsf{Q}_P$, a contradiction.

Building the extensions: from groups to girales

Now we're going to build some girales out of the members of the Q_P's.

For each abelian group $\mathbf{G}$, we define a lattice ordered algebra $R(\mathbf{G})$ by thinking of $\mathbf{G}$ as discretely ordered and adding a top $\top$ and bottom $\bot$.

Multiplication is extended from $\mathbf{G}$ by defining $a \cdot \top = \top \cdot a = \top$ for $a \neq \bot$, and $a \cdot \bot = \bot \cdot a = \bot$. The unit of $\mathbf{G}$ is also a unit for $R(\mathbf{G})$.

Residuals are given as usual: $a \to c = \max\{b \mid ab \leq c\}$. Also, the unit of $\mathbf{G}$ is a negation constant: $(a \to 1) \to 1 = a$.

Finally, we define $!a = a \wedge 1$.

Building the extensions: distinct varieties

Now define $\mathsf{K}_P = \mathbb{I}(\{R(\mathbf{G}) \mid \mathbf{G} \in \mathsf{Q}_P\})$ and $\mathsf{V}_P = \mathbb{V}(\mathsf{K}_P)$. Note that K_P is a universal class defined by Σ_P and

$$(\forall x)((x \not\approx \bot \& (x \not\approx \top) \implies (x(x \to 1) \approx 1)), \quad (1)$$

$$(\forall x)(\forall y)((x \not\approx \bot)\&(y \not\approx \bot)\&(x \not\approx y) \implies (x \vee y \approx \top)), \quad (2)$$

$$(\forall x)(\forall y)((x \not\approx \top)\&(y \not\approx \top)\&(x \not\approx y) \implies (x \wedge y \approx \bot)), \quad (3)$$

$$(\forall x)((x \not\approx \bot) \implies (x \cdot \top \approx \top)). \quad (4)$$

Thus:

Lemma:

1. $\mathbb{HSP}_{\mathsf{U}}('K_P) = \mathsf{K}_P$.
2. $\mathsf{V}_P = \mathbb{ISP}(\mathsf{K}_P)$ and the class of finitely subdirectly irreducible members of V_P is exactly K_P.
3. If P, P' are distinct non-empty sets of primes, then $\mathsf{V}_P \neq \mathsf{V}_{P'}$.

The main theorem

Theorem:

For each non-empty set of primes P, V_P has the amalgamation property. Hence, the corresponding logic has the deductive interpolation property, and there are continuum many of these.

Some final remarks

- We can also show that there are uncountably many extensions without the DIP.
- None of the extensions with DIP have the CIP, but they have a weak form of the CIP
- It remains open whether there are continuum-many extensions with the CIP. Also, it is open whether we can get these results with weakening added (stick around for the next talk).
- Probably could not have done this proof theoretically: This is a testament to algebraic methods in linear logic as well as the fact that linear logic is a modal logic.

Thank you!

You can find more details at

W. Fussner and S. Santschi, Interpolation in Linear Logic and Related Systems, to appear in ACM Transactions on Computational Logic, `https://arxiv.org/abs/2305.05051`.

See also:

P. Aglianò, An algebraic investigation of linear logic, `https://arxiv.org/abs/2305.12408`.

Thank you!