Interpolation in Extensions of Linear Logic

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TLLA 2024
Tallinn, Estonia

8 July 2024

Interpolation: a brief history

Various kinds of interpolation properties have been studied quite a lot:

- Just seven consistent superintuitionistic logics with CIP/DIP (Maksimova)
- At most 49 consistent normal extensions of **S4** with DIP (Maksimova)
- CIP for basic substructural logics like **FL**, $\mathbf{FL_e}$, $\mathbf{FL_c}$, **LL**, **MALL**, etc
- Countably infinitely many extensions of Łukasiewicz's infintie valued logic with DIP (Di Nola–Lettieri)

Interpolation: main themes

Big question: How do we make sense of this zoo of different results?

Convention wisdom: Interpolation is a rather uncommon property

Main theorem:

There are continuum-many axiomatic extensions of each of $\mathbf{FL_e}$, **LL**, and **MALL** with the deductive interpolation property, and this remains true for many modifications of the basic language (e.g. adding or deleting bounds).

The proof is inherently algebraic and is probably hard to simulate with other methods.

Interpolation: variants

Craig interpolation:

$$A \to B \Rightarrow \exists I(\mathrm{var}(I) \subseteq \mathrm{var}(A) \cap \mathrm{var}(B), A \to I, I \to B)$$

Deductive interpolation:

$$A \vdash B \Rightarrow \exists I(\mathrm{var}(I) \subseteq \mathrm{var}(A) \cap \mathrm{var}(B), A \vdash I, I \vdash B)$$

Other variations: Lyndon interpolation, uniform interpolation, etc.

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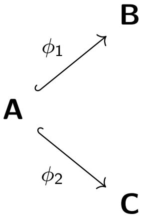
Interpolation done algebraically

Theorem (Czelakowski-Dziobiak):

Let $\vdash$ be an algebraizable deductive system with a local deduction theorem and equivalent algebraic semantics V. Then $\vdash$ has the DIP if and only if V has the amalgamation property.



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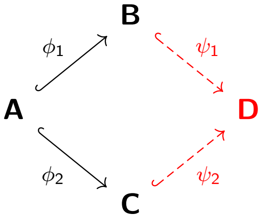

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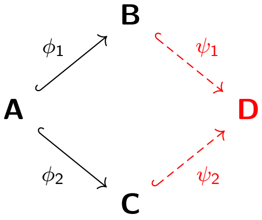


Equational consequence

- **Algebra**: A structure $\mathbf{A}$ consisting of a set and some functions/operations $A^n \to A$, each with an arity n.
- **Variety**: A class of algebras defined by equations — $\forall \vec{x}[s(\vec{x}) \approx t(\vec{x})]$.
- **Equation consequence** of a class K: Relation $\models_{\mathsf{K}}$ between sets of equations and equations given by

$$E \models_{\mathsf{K}} (u \approx w) \iff \text{For each } \mathbf{A} \in \mathsf{K} \text{ and each assignment } h \text{ into } \mathbf{A}, h(u) = h(w) \text{ whenever } h(s) = h(t) \text{ for all } (s \approx t) \in E.$$

Algebraization

A deductive system $\vdash$ is algebraizable if it there are mutually inverse translations between $\vdash$ and $\models_{\mathsf{V}}$ for some variety V.

Then V is the equivalent algebraic semantics of $\vdash$. This is much stronger than just being a sound and complete algebraic semantics for $\vdash$.

Proposition (Blok-Pigozzi):

Let $\vdash$ be an algebraizable deductive system and V be its equivalent algebraic semantics. Then the lattice of axiomatic extensions of $\vdash$ is dually isomorphic to the lattice of subvarieties of V.

There are well known algebraic semantics for **LL**, but P. Aglianò gave an equivalent algebraic semantics for **LL** in the 1990s that was never published.

Aglianò's semantics: girales

Written using notation from substructural logic, a girale is an algebra of the form $\langle A, \wedge, \vee, \cdot, \rightarrow, 0, 1, \bot, \top, ! \rangle$ such that

1. $\langle A, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice;
2. $\langle A, \cdot, 1 \rangle$ is a commutative monoid;
3. for all $x, y, z \in A$,

$$x \cdot y \leq z \iff x \leq y \rightarrow z;$$

4. For all $x \in A$, $x = (x \rightarrow 0) \rightarrow 0$.
5. for all $x \in A$:
   - $!(x \wedge y) = !x \cdot !y$
   - $!!x = !x \leq x \wedge 1$
   - $!1 = 1$

Both ? and $\invamp$ are definable from the above.

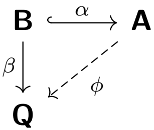
Building the extensions: roadmap

To construct continuum-many axiomatic extensions of **LL** with DIP, we proceed by

1. constructing continuum-many classes of abelian groups that satisfy some well-tailored Horn formulas,
2. using the categorical algebra of abelian groups to show that these have amalgamation,
3. using these groups to define continuum-many distinct varieties of girales with amalgamation,
4. and transfering amalgamation for these to DIP for the corresponding extensions of **LL**.

Building the extensions: injectives

An algebra $\mathbf{Q}$ in a class K is called injective if for all $\mathbf{A}, \mathbf{B} \in$ K, every embedding $\alpha\colon \mathbf{B} \to \mathbf{A}$, and every homomorphism $\beta\colon \mathbf{B} \to \mathbf{Q}$, there exists $\varphi\colon \mathbf{A} \to \mathbf{Q}$ with $\varphi \circ \alpha = \beta$:



A class K has enough injectives if every algebra in K embeds into an algebra in K that is injective over K)

Lemma (folklore):

Suppose K is a class of similar algebras that is closed under finite products. If K has enough injectives, then K has the amalgamation property.

Building the extensions: starting from abelian groups

A subgroup **G** of a group **H** is an essential subgroup of **H** if for every non-trivial subgroup **G**$'$ of **H** we have that $G \cap G'$ is non-trivial.

Lemma (Eckmann–Schopf 1953):

Every abelian group is an essential subgroup of an injective abelian group.

For each set P of primes numbers we define a set of quasiequations by

$$\Sigma_P = \{x^p \approx 1 \Rightarrow x \approx 1 \mid p \in P\}.$$

The quasivariety of abelian groups defined by Σ_P is called Q_P.

Building the extensions: starting from abelian groups

Lemma:

For every set of prime numbers P, the quasivariety Q_P has the amalgamation property.

Proof: It's enough to show that Q_P has enough injectives.

Let $\mathbf{G} \in \mathsf{Q}_P$. By the lemma, $\mathbf{G}$ is an essential subgroup of an injective abelian group $\mathbf{H}$. it suffices to show that $\mathbf{H} \in \mathsf{Q}_P$.

Toward a contradiction, suppose $\mathbf{H} \notin \mathsf{Q}_P$. Then there is $p \in P$ and $a \in H$ with $a^p = 1$ and $a \neq 1$.

The subgroup $\mathbf{S}$ generated by a is cyclic of order p and $\mathbf{G}$ is an essential subgroup of $\mathbf{H}$, so there exists $b \in S \cap G$ with $b \neq 1$. But then $b^p = 1$, so $\mathbf{G} \notin \mathsf{Q}_P$, a contradiction.

Now we're going to build some girales out of the members of the Q_P's.

For each abelian group **G**, we define a lattice ordered algebra $R(\mathbf{G})$ by thinking of **G** as discretely ordered and adding a top $\top$ and bottom $\bot$.

Multiplication is extended from **G** by defining $a \cdot \top = \top \cdot a = \top$ for $a \neq \bot$, and $a \cdot \bot = \bot \cdot a = \bot$. The unit of **G** is also a unit for $R(\mathbf{G})$.

Residuals are given as usual: $a \to c = \max\{b \mid ab \leq c\}$. Also, the unit of **G** is a negation constant: $(a \to 1) \to 1 = a$.

Finally, we define $!a = a \wedge 1$.

Building the extensions: distinct varieties

Now define $\mathsf{K}_P = \mathbb{I}(\{R(\mathbf{G}) \mid \mathbf{G} \in \mathsf{Q}_P\})$ and $\mathsf{V}_P = \mathbb{V}(\mathsf{K}_P)$. Note that K_P is a universal class defined by Σ_P and

$$(\forall x)((x \not\approx \bot \& (x \not\approx \top) \implies (x(x \to 1) \approx 1)), \quad (1)$$

$$(\forall x)(\forall y)((x \not\approx \bot)\&(y \not\approx \bot)\&(x \not\approx y) \implies (x \vee y \approx \top)), \quad (2)$$

$$(\forall x)(\forall y)((x \not\approx \top)\&(y \not\approx \top)\&(x \not\approx y) \implies (x \wedge y \approx \bot)), \quad (3)$$

$$(\forall x)((x \not\approx \bot) \implies (x \cdot \top \approx \top)). \quad (4)$$

Thus:

Lemma:

1. $\mathbb{HSP}_{\mathsf{U}}(\mathsf{K}_P) = \mathsf{K}_P$.
2. $\mathsf{V}_P = \mathbb{ISP}(\mathsf{K}_P)$ and the class of finitely subdirectly irreducible members of V_P is exactly K_P.
3. If P, P' are distinct non-empty sets of primes, then $\mathsf{V}_P \neq \mathsf{V}_{P'}$.

The main theorem

Theorem:

For each non-empty set of primes P, V_P has the amalgamation property. Hence, the corresponding logic has the deductive interpolation property, and there are continuum many of these.

Some final remarks

- None of the extensions with DIP have the CIP, but they have a weak form of the CIP
- It remains open whether there are continuum-many extensions with the CIP. Also, it is open whether we can get these results with weakening added.
- Best understood in contrast with the case without the exchange rule.

Thank you!

You can find more details at

W. Fussner and S. Santschi, Interpolation in Linear Logic and Related Systems, to appear in ACM Transactions on Computational Logic, https://arxiv.org/abs/2305.05051.

See also:

P. Aglianò, An algebraic investigation of linear logic, https://arxiv.org/abs/2305.12408.

Thank you!